ON SUBSPACES OF NON-REFLEXIVE ORLICZ SPACES

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ABSTRACT. Kadec and Pelczýnski have shown that every non-reflexive subspace of $L^1(\mu)$ contains a copy of l_1 complemented in $L^1(\mu)$. On the other hand Rosenthal investigated the structure of reflexive subspaces of $L^1(\mu)$ and proved that such subspaces, have non-trivial type. We show the same facts to hold true, for a special class of non-reflexive Orlicz spaces. In particular we show that *if* F *is an N-function in* Δ_2 *with its complement* G *satisfying* $\lim_{t\to\infty} \frac{G(ct)}{G(t)} = \infty$ *then every non-reflexive subspace of* L_F^*, *contains a copy of* l_1 *complemented in* L_F^*. Furthermore we establish the fact that *if* F *is an N-function in* Δ_2 *with its complement* G *satisfying* $\lim_{t\to\infty} \frac{G(ct)}{G(t)} = \infty$ *then every reflexive subspace of* L_F^* *has non trivial type.*

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1 Introduction and Background

Kadec and Pelczýnski in [4] have shown that every non-reflexive subspace of $L^1(\mu)$ contains a copy of l_1 complemented in $L^1(\mu)$. On the other hand Rosenthal in [12] investigated the structure of reflexive subspaces of $L^1(\mu)$ and proved that such subspaces, have non-trivial type.

In this paper we will establish similar results for a special class of non-reflexive Orlicz spaces. In particular, in Section 2 we show that *if F is an N-function in Δ_2 with its complement G satisfying* $\lim_{t\to\infty} \frac{G(ct)}{G(t)} = \infty$ *then every non-reflexive subspace of L_F^*, contains a copy of l_1 complemented in L_F^* (Theorem 2.4).* Furthermore we show that *if F is an N-function in Δ_2 with its complement G satisfying* $\lim_{t\to\infty} \frac{G(ct)}{G(t)} = \infty$ *then every reflexive subspace of L_F^* has non trivial type (Theorem 2.7).*

1.1 N-Functions and Orlicz Spaces

We begin with recalling some basic facts about N-functions and Orlicz Spaces. For a detailed account of these facts, the reader could consult chapters one and two in [5]. Throughout this paper μ denotes a probability.

Definition 1.1 *Let $p : [0, \infty) \to [0, \infty)$ be a right continuous, monotone increasing function with*

1. $p(0) = 0$;
2. $\lim_{t\to\infty} p(t) = \infty$;
3. $p(t) > 0$ *whenever* $t > 0$;

then the function defined by

$$F(x) = \int_0^{|x|} p(t)dt$$

is called an N-function.

The following proposition gives an alternative view of N-functions.

Proposition 1.1 *The function F is an N-function if and only if F is continuous, even and convex with*

1. $\lim_{x\to 0} \frac{F(x)}{x} = 0$;
2. $\lim_{x\to\infty} \frac{F(x)}{x} = \infty$;
3. $F(x) > 0$ *if* $x > 0$.

Definition 1.2 *For an N-function F define*

$$G(x) = \sup\{t|x| - F(t) : t \geq 0\} .$$

Then G is an N-function and it is called the complement of F.

Observe that F is the complement of its complement G.

Definition 1.3 *An N-function F is said to satisfy the Δ_2 condition ($F \in \Delta_2$) if* $\limsup_{x\to\infty} \frac{F(2x)}{F(x)} < \infty$. *That is, there is a $K > 0$ so that $F(2x) \leq KF(x)$ for large values of x.*

Given an N-function F, the corresponding space of F-integrable functions is defined as follows.

Definition 1.4 *For an N-function F and a measurable f define*

$$\mathbf{F}(f) = \int F(f) d\mu.$$

Let $L_F = \{f \text{ measurable} : \mathbf{F}(f) < \infty\}$. If G denotes the complement of F let

$$L_F^* = \{f \text{ measurable} : |\int fg d\mu| < \infty \quad \forall g \in L_G\} .$$

The collection L_F^* is then a linear space. For $f \in L_F^*$ define

$$\|f\|_F = \sup\{|\int fg d\mu| : \mathbf{G}(g) \leq 1\} .$$

Then $(L_F^*, \|\cdot\|_F)$ is a Banach space, called an Orlicz space. Moreover, letting $\|\cdot\|_{(F)}$ be the Minkowski functional associated with the convex set $\{f \in L_F^* \ : \ \mathbf{F}(f) \leq 1\}$, we have that $\|\cdot\|_{(F)}$ is an equivalent norm on L_F^*, called the Luxemburg norm. Indeed, $\|f\|_{(F)} \leq \|f\|_F \leq 2\|f\|_{(F)}$, for all $f \in L_F^*$.

The following theorem establishes the fact that an Orlicz space is a dual space.

Theorem 1.2 *Let F be an N-function and let E_F be the closure of the bounded functions in L_F^*. Then the conjugate space of $(E_F, \|\cdot\|_{(F)})$ is $(L_G^*, \|\cdot\|_G)$, where G is the complement of F.*

Theorem 1.3 *Let F be an N-function and G be its complement. Then the following statements are equivalent:*

1. $L_F^* = E_F$.

2. $L_F^* = L_F$.

3. The dual of $(E_F, \|\cdot\|_{(F)})$ *is* $(L_G^*, \|\cdot\|_G)$.

4. $F \in \Delta_2$.

Theorem 1.4 (Hölder's Inequality) *For* $f \in L_F^*$ *and* $g \in L_G^*$ *we have*

$$\int |fg| d\mu \leq \|f\|_F \cdot \|g\|_{(G)} \, .$$

Theorem 1.5 *If* $f \in L_F^*$ *then*

$$\|f\|_F = \inf\left\{\frac{1}{k}(1 + \mathbf{F}(kf)) : k > 0\right\}.$$

It follows then that $f \in L_F^*$ *if and only if there is* $c > 0$ *so that* $\mathbf{F}(cf) < \infty$.

Proposition 1.6 *If* $\|f\|_F \leq 1$ *then* $f \in L_F$ *and* $\mathbf{F}(f) \leq \|f\|_F$.

Recall that a subset $\mathcal{K}$ of $L^1(\mu)$ is called uniformly integrable if given $\varepsilon > 0$ there is a $\delta > 0$ so that $\sup\left\{\int_E |f| d\mu : f \in \mathcal{K}\right\} < \varepsilon$ whenever $\mu(E) < \delta$. Alternatively $\mathcal{K}$ is bounded and uniformly integrable if and only if given $\varepsilon > 0$ there is an $N > 0$ so that

$$\sup\left\{\int_{[\,|f|>c\,]} |\, f \,| \, d\mu : f \in \mathcal{K}\right\} < \varepsilon \text{ whenever } c \geq N.$$

The classical theorem of Dunford and Pettis [2, page 93], identifies the bounded, uniformly integrable subsets of $L^1(\mu)$ with the relatively weakly compact sets. A concept similar to Uniform Integrability is that of equi-absolute continuity.

Definition 1.5 *We say that a collection* $\mathcal{K} \subset L_F^*$ *has equi-absolutely continuous norms if*

$$\forall\, \varepsilon > 0 \,\exists\, \delta > 0 \text{ so that } \sup\{\|\chi_E f\|_F : f \in \mathcal{K}\} < \varepsilon \text{ whenever } \mu(E) < \delta.$$

For $f \in L_F^*$ *we say that* f *has absolutely continuous norm if* $\{f\}$ *has equi-absolutely continuous norms.*

The following result deal with the equi-absolute continuity of the norms.

Theorem 1.7 *A function* $f \in L_F^*$ *has absolutely continuous norm if and only if* $f \in E_F$.

The next two results resemble the theorem of Dunford and Pettis. For their proofs the reader should consult [1], Lemma 2.1 and Corollary 2.9.

Theorem 1.8 *If $F \in \Delta_2$ and $\mathcal{K} \subset L_F^*$ then the following statements are equivalent:*

I) The set $\mathcal{K}$ has equi-absolutely continuous norms.

II)The collection $\{F(f) : f \in \mathcal{K}\}$ is uniformly integrable in L^1.

Theorem 1.9 *Let $F \in \Delta_2$ and suppose that its complement G satisfies*

$$\lim_{t\to\infty} \frac{G(ct)}{G(t)} = \infty \ for \ some \ c > 0 \, .$$

Then a bounded set $\mathcal{K} \subset L_F^*$ is relatively weakly compact if and only if $\mathcal{K}$ has equi-absolutely continuous norms.

1.2 Banach Spaces with Type

Denote by (r_n), the sequence of Rademacher functions. Recall that for a positive integer n, $r_n : [0,1] \to \{-1,1\}$ is defined by

- $r_n(1) = -1$.
- $r_n(t) = (-1)^{(i-1)}$ for $t \in [\frac{i-1}{2^n}, \frac{i}{2^n})$, where $i = 1, \ldots, 2^n$.

Definition 1.6 *A Banach space X is said to have type p, for some $1 < p \leq 2$, if there is a constant K so that*

$$\left(\int_0^1 \|\sum_{i=1}^n r_i(t)x_i\|^p dt\right)^{\frac{1}{p}} \leq K\left(\sum_{i=1}^n \|x_i\|^p\right)^{\frac{1}{p}} ,$$

for any $x_1, \ldots, x_n \in X$.

It turns out that type's presence in a Banach space, is ultimately connected with the space's finite dimensional structure. To be more specific, we need the following notion.

Definition 1.7 *Let $\lambda \geq 1$ and X be a Banach space. We say that X contains l_1^n's λ-uniformly if for each positive integer n there is an isomorphism $T : l_1^n \to X$ so that $\|T\| \cdot \|T^{-1}\| \leq \lambda$.*

It is easy to see from the definition above that X contains l_1^n's λ-uniformly if and only if for each positive integer n, $\exists\, x_1, \ldots, x_n \in B_X$ such that

$$\|\sum_{i=1}^n a_i x_i\| \geq \frac{1}{\lambda}\sum_{i=1}^n |a_i| ,$$

for all choices of scalars $a_1, \ldots, a_n$.

Theorem 1.10 (Pisier) *The following are equivalent for a Banach space X :*

1. *For each $\lambda > 1$, X does not contain l_1^n's λ-uniformly.*

2. *For some $\lambda > 1$, X does not contain l_1^n's λ-uniformly.*

3. *The space X has type p for some $1 < p \leq 2$.*

For a proof of this theorem as well as a more detailed account and bibliography, the reader should consult [11] and [10, pages 31-40].

2 The Main Results

2.1 Subspaces containing complemented l_1

In this section we derive a theorem similar to the one of Kadec and Pelczýnski, about L^1 in [4] (see also [2, pages 94-98]).

Lemma 2.1 *Let (f_n) be a normalized disjointly supported sequence in L_F^*, where $F \in \Delta_2$ and its complement G satisfies $\lim_{x\to\infty} \frac{G(cx)}{G(x)} = \infty$, for some $c > 0$. Then there is a subsequence (f_{n_k}) of (f_n) so that*

i. *(f_{n_k}) is equivalent to l_1's unit vector basis.*

ii. *The closed linear span of (f_{n_k}) is complemented in L_F^* by means of a projection of norm less than or equal to $4c$.*

iii. *The coefficient functionals (ϕ_k) extend to all of the dual of L_F^* and $\|\phi_k\| \leq 4$ for all positive integers k.*

Proof : Let E_n denote the support of f_n. For each positive integer n choose $g_n \in L_G$ with $\int G(g_n)d\mu \leq 1$ so that $\int g_n f_n d\mu \geq \frac{1}{2}$. There is no harm in assuming that each g_n is also supported on E_n.

Claim that $\int G(g_n/c)d\mu \to 0$ as $n \to \infty$. Fix $\varepsilon > 0$. Since $\lim_{x\to\infty} \frac{G(cx)}{G(x)} = \infty$ then $\lim_{x\to\infty} \frac{G(x/c)}{G(x)} = 0$. So we can choose $x_0 > 0$ so that $\frac{G(x/c)}{G(x)} < \frac{\varepsilon}{2}$ whenever $x \geq x_0$. Since the E_n's are pairwise disjoint and μ is a probability, we have that $\mu(E_n) \to 0$ as $n \to \infty$. So there is a positive integer N so that $\mu(E_n) < \frac{\varepsilon}{2G(x_0/c)}$ whenever $n \geq N$. So for $n \geq N$ we have

$$\int G(g_n/c)d\mu \quad = \quad \int_{[\,|g_n|<x_0\,]} G(g_n/c)d\mu + \int_{[\,|g_n|\geq x_0\,]} G(g_n/c)d\mu$$

$$
\begin{aligned}
&\leq \; G(x_0/c)\mu(E_n) + \frac{\varepsilon}{2}\int G(g_n)d\mu \\
&< \; \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon
\end{aligned}
$$

and so the claim is established.

Now choose a subsequence (n_k) of the positive integers so that $\sum_{k=1}^{\infty}\int G(\frac{g_{n_k}}{c})d\mu \leq 1$. For any sequence of signs $\sigma = (\varepsilon_k)$ define $g_\sigma = \sum_{k=1}^{\infty}\varepsilon_k g_{n_k}$. Since the g_{n_k}'s are disjointly supported, g_σ is well defined. Furthermore

$$
\begin{aligned}
\int G(\frac{g_\sigma}{c})d\mu &= \sum_{k=1}^{\infty}\int_{E_{n_k}} G(\frac{g_\sigma}{c})d\mu \\
&= \sum_{k=1}^{\infty}\int_{E_{n_k}} G(\frac{\varepsilon_k g_{n_k}}{c})d\mu \\
&= \sum_{k=1}^{\infty}\int_{E_{n_k}} G(\frac{g_{n_k}}{c})d\mu \\
&\leq 1 \, .
\end{aligned}
$$

So $g_\sigma \in L_G^*$. Recall that the norm of g_σ in L_G^*, is given by $\|g_\sigma\|_G = \inf\{\frac{1}{k}(1+\int G(kg_\sigma)d\mu) : k > 0\}$ and so it is easy to see that $\|g_\sigma\|_G$ remains constant as σ varies. Denote this constant by M and observe that

$$
M = \|g_\sigma\|_G \leq c(1+\int G(\frac{g_\sigma}{c})d\mu) = c(1+\sum_{k=1}^{\infty}\int G(\frac{g_{n_k}}{c})d\mu) \leq 2c \, .
$$

Now for $(a_k) \in l_1$ let $\sigma = (sign(a_k))$. Then

$$
\begin{aligned}
\|\sum_{k=1}^{\infty} a_k f_{n_k}\|_F &\geq \frac{1}{\|g_\sigma\|_G}\int (g_\sigma \sum_{k=1}^{\infty} a_k f_{n_k})d\mu \\
&= \frac{1}{M}\int (\sum_{k=1}^{\infty} |a_k| g_{n_k} f_{n_k})d\mu \\
&= \frac{1}{M}\sum_{k=1}^{\infty}|a_k|\int g_{n_k} f_{n_k} d\mu \\
&\geq \frac{1}{2M}\sum_{k=1}^{\infty}|a_k| \, .
\end{aligned}
$$

Hence (i) is established.

Now define for each k, a functional ϕ_k on all of L_F^* by

$$
\phi_k(f) = \frac{1}{\int g_{n_k} f_{n_k} d\mu}\cdot\int g_{n_k} f d\mu
$$

and define $P : L_F^* \to L_F^*$ by

$$P(f) = \sum_{k=1}^{\infty} \phi_k(f) f_{n_k}.$$

Then for $k = 1, 2, \ldots$

$$\|\phi_k\| \leq 2\|g_{n_k}\|_G \leq 2 \cdot (1 + \mathbf{G}(g_{n_k})) \leq 4 \, .$$

Furthermore P is a projection of L_F^* onto the closed linear span of (f_{n_k}) with

$$\begin{aligned}
\|P\| &= \sup_{\|f\|_F \leq 1} \|\sum_{k=1}^{\infty} \frac{\int g_{n_k} f d\mu}{\int g_{n_k} f_{n_k} d\mu} \cdot f_{n_k}\|_F \\
&\leq 2 \sup_{\|f\|_F \leq 1} \sum_{k=1}^{\infty} \int |g_{n_k} f| d\mu \\
&\leq 2 \sup_{\|f\|_F \leq 1} \|(\sum_{k=1}^{\infty} |g_{n_k}|)\|_G \cdot \|f\|_F \\
&= 2M \\
&\leq 4c.
\end{aligned}$$

And so our proof is complete. ∎

We state now the following result in form of a lemma. Its proof can be found in [2, page 50].

Lemma 2.2 *Let (z_n) be a basic sequence in the Banach space X with coefficient functionals (z_n^*). Suppose that there is a bounded linear projection $P : X \to X$ onto the closed linear span $[z_n]$ of (z_n). If (y_n) is any sequence in X for which*

$$\sum_{n=1}^{\infty} \|P\| \cdot \|z_n^*\| \cdot \|z_n - y_n\| < 1,$$

then (y_n) is a basic sequence equivalent to (z_n) and the closed linear span $[y_n]$ of (y_n) is also complemented in X.

Lemma 2.3 *Let (f_n) be a sequence in L_F^* where $F \in \Delta_2$ and its complement G satisfies $\lim_{x\to\infty} \frac{G(cx)}{G(x)} = \infty$ for some $c > 0$. Suppose that for each $\varepsilon > 0$ there is a positive integer n_ε so that $\mu([\,|f_{n_\varepsilon}| \geq \varepsilon\|f_{n_\varepsilon}\|_F\,]) < \varepsilon$. Then there is a subsequence (r_n) of (f_n) so that $(\frac{r_n}{\|r_n\|_F})$ is equivalent to l_1's unit vector basis. Furthermore the closed linear span $[r_n]$ of (r_n) is complemented in L_F^*.*

Proof : First observe that if $f \in L_F^*$, $E = [\,|f| \geq \varepsilon\|f\|_F\,]$ and K is the norm of the inclusion map $L_G^* \hookrightarrow L^1$ then

$$\|\chi_E \frac{f}{\|f\|_F}\|_F \geq \|\frac{f}{\|f\|_F}\|_F - \|\chi_{E^c} \frac{f}{\|f\|_F}\|_F$$

$$
\begin{aligned}
&= 1-\frac{1}{\|f\|_F}\sup\{|\int g\chi_{E^c} f d\mu| : g\in L_G \ \mathit{and}\ \mathbf{G}(g)\le 1\} \\
&\ge 1-\frac{1}{\|f\|_F}\sup\{\|g\|_1\cdot\|\chi_{E^c}f\|_\infty : g\in L_G \ \mathit{and}\ \mathbf{G}(g)\le 1\} \\
&\ge 1-\frac{K}{\|f\|_F}\|\chi_{E^c}f\|_\infty \\
&\ge 1-\frac{K}{\|f\|_F}\|f\|_F\cdot\varepsilon \\
&= 1-K\varepsilon\ .
\end{aligned}
$$

So using the hypothesis there is a measurable set E_1 and a positive integer n_1 so that

$$\mu(E_1)<\frac{1}{16c\cdot 4^2K}\ \ \mathit{and}\ \ \|\chi_{E_1}\frac{f_{n_1}}{\|f_{n_1}\|_F}\|_F\ge 1-\frac{1}{16c\cdot 4^2}\ .$$

Since $F\in\Delta_2$ then each $f\in L_F^*$ has an absolutely continuous norm. This fact together with the hypothesis again, yields a measurable E_2 and a positive integer $n_2>n_1$ so that

$$\mu(E_2)<\frac{1}{16c\cdot 4^3K},$$

$$\|\chi_{E_2}\frac{f_{n_2}}{\|f_{n_2}\|_F}\|_F>1-\frac{1}{16c\cdot 4^3}$$

and

$$\|\chi_{E_2}\frac{f_{n_1}}{\|f_{n_1}\|_F}\|_F<\frac{1}{16c\cdot 4^3}\ .$$

Continue inductively to construct a subsequence (g_n) of (f_n) and a sequence of measurable sets (E_n) so that

$$\mu(E_n)<\frac{1}{16c\cdot 4^{n+1}K},$$

$$\|\chi_{E_n}\frac{g_n}{\|g_n\|_F}\|_F>1-\frac{1}{16c\cdot 4^{n+1}}$$

and

$$\sum_{k=1}^{n-1}\|\chi_{E_n}\frac{g_k}{\|g_k\|_F}\|_F<\frac{1}{16c\cdot 4^{n+1}}\ .$$

Now let

$$A_n=E_n\setminus\bigcup_{k=n+1}^{\infty}E_k\ \ \mathit{and}\ \ h_n=\frac{g_n}{\|g_n\|_F}\chi_{A_n}\ .$$

Then

$$
\begin{aligned}
\|\frac{g_n}{\|g_n\|_F}-h_n\|_F &= \|\chi_{A_n^c}\frac{g_n}{\|g_n\|_F}\|_F \\
&\le \|\chi_{E_n^c}\frac{g_n}{\|g_n\|_F}\|_F+\|\chi_{E_n\setminus A_n}\frac{g_n}{\|g_n\|_F}\|_F
\end{aligned}
$$

$$
\begin{aligned}
&\leq \frac{1}{16c\cdot 4^{n+1}} + \|\chi_{\bigcup_{k=n+1}^{\infty} E_k}\frac{g_n}{\|g_n\|_F}\|_F \\
&\leq \frac{1}{16c\cdot 4^{n+1}} + \|\sum_{k=n+1}^{\infty}\chi_{E_k}\frac{g_n}{\|g_n\|_F}\|_F \\
&\leq \frac{1}{16c\cdot 4^{n+1}} + \sum_{k=n+1}^{\infty}\|\chi_{E_k}\frac{g_n}{\|g_n\|_F}\|_F \\
&\leq \frac{1}{16c\cdot 4^{n+1}} + \sum_{k=n+1}^{\infty}\frac{1}{16c\cdot 4^{k+1}} \\
&< \frac{1}{16c\cdot 4^{n}}\ .
\end{aligned}
$$

Thus

$$
\begin{aligned}
1 &\geq \|h_n\|_F \\
&= \|\chi_{A_n}\frac{g_n}{\|g_n\|_F}\|_F \\
&\geq \|\chi_{E_n}\frac{g_n}{\|g_n\|_F}\|_F - \|\chi_{\bigcup_{k=n+1}^{\infty} E_k}\frac{g_n}{\|g_n\|_F}\|_F \\
&\geq 1 - \frac{1}{16c\cdot 4^{n+1}} - \sum_{k=n+1}^{\infty}\|\chi_{E_k}\frac{g_n}{\|g_n\|_F}\|_F \\
&\geq 1 - \frac{1}{16c\cdot 4^{n+1}} - \sum_{k=n+1}^{\infty}\frac{1}{16c\cdot 4^{k+1}} \\
&> 1 - \frac{1}{16c\cdot 4^{n}}\ .
\end{aligned}
$$

And so

$$
\begin{aligned}
\|\frac{g_n}{\|g_n\|_F} - \frac{h_n}{\|h_n\|_F}\|_F &\leq \|\frac{g_n}{\|g_n\|_F} - h_n\|_F + \|h_n - \frac{h_n}{\|h_n\|_F}\|_F \\
&\leq \frac{1}{16c\cdot 4^{n}} + (1 - \|h_n\|_F) \\
&\leq \frac{1}{16c\cdot 4^{n}} + (1 - 1 + \frac{1}{16c\cdot 4^{n}}) \\
&= \frac{2}{16c\cdot 4^{n}}\ .
\end{aligned}
$$

By Lemma (2.1), there is a subsequence (n_k) of the positive integers so that

- $(\frac{h_{n_k}}{\|h_{n_k}\|_F})$ is equivalent to l_1's unit vector basis.
- The closed linear span $[h_{n_k}]$ of (h_{n_k}) is complemented in L_F^* by means of a projection P, of norm less than or equal to $4c$.
- The coefficient functionals ϕ_k extend to all of L_G^* with $\|\phi_k\|_G \leq 4$ for all k.

So we have that if $r_k = g_{n_k}$ then

$$\begin{aligned}
\sum_{k=1}^{\infty} \|P\| \cdot \|\phi_k\|_G \cdot \|\frac{r_k}{\|r_k\|_F} - \frac{h_{n_k}}{\|h_{n_k}\|_F}\|_F &\leq 16c \cdot \sum_{k=1}^{\infty} \|\frac{g_{n_k}}{\|g_{n_k}\|_F} - \frac{h_{n_k}}{\|h_{n_k}\|_F}\|_F \\
&\leq 16c \cdot \sum_{n=1}^{\infty} \|\frac{g_n}{\|g_n\|_F} - \frac{h_n}{\|h_n\|_F}\|_F \\
&\leq 16c \cdot \sum_{n=1}^{\infty} \frac{2}{16c \cdot 4^n} \\
&= \sum_{n=1}^{\infty} \frac{2}{4^n} \\
&< 1 .
\end{aligned}$$

Hence the result is established by an appeal to Lemma (2.2). ∎

Theorem 2.4 *Let $F \in \Delta_2$ with its complement G satisfying*

$$\lim_{x \to \infty} \frac{G(cx)}{G(x)} = \infty \quad for \quad some \quad c > 0.$$

If X is any non-reflexive subspace of L_F^* then X contains an isomorphic copy of l_1 that is complemented in L_F^*.

Proof : Since X is not reflexive, then the ball B_X of X is not relatively weakly compact. Hence by Theorem (1.9), B_X does not have equi-absolutely continuous norms. So by Theorem (1.8), the set $\{F(f) \,:\, f \in B_X\}$ is not uniformly integrable in L^1. Thus there is a $\delta > 0$ so that

$$\lim_{a \to \infty} \sup\{\int_{[\,|f| \geq a\,]} F(f) d\mu \,;\; f \in B_X\} = \delta \,.$$

Keeping in mind that the above limit is actually an infimum we can find an increasing sequence (a_n) of positive reals, with $a_n \to \infty$ as $n \to \infty$ so that

$$\delta \leq \sup\{\int_{[\,|f| \geq a_n\,]} F(f) d\mu \,;\; f \in B_X\} < \delta + \frac{1}{n} \,,$$

for each positive integer n. It follows then, that there is a sequence (f_n) in B_X so that

$$\delta - \frac{1}{n} < \int_{[\,|f_n| \geq a_n\,]} F(f_n) d\mu < \delta + \frac{1}{n}$$

for all positive integers n. Now let $g_n = f_n \chi_{[\,|f_n| \geq a_n\,]}$ and $h_n = f_n - g_n$. Observe that for each $\varepsilon > 0$ we have

$$\mu([\,|g_n| \geq \varepsilon \|g_n\|_F\,]) \quad \leq \quad \mu([\,|g_n| > 0\,])$$

$$
\begin{aligned}
&\le \mu([\,|f_n| \ge a_n\,]) \\
&\le \frac{1}{a_n}\int_{[\,|f_n|\ge a_n\,]} |f_n| d\mu \\
&\le \frac{1}{a_n}\int_{[\,|f_n|\ge a_n\,]} F(f_n) d\mu \\
&\le \frac{1}{a_n}\,,
\end{aligned}
$$

provided that n is large enough. Since $\frac{1}{a_n} \to 0$ as $n \to \infty$ then $\mu([\,|g_n| \ge \varepsilon \|g_n\|_F\,]) < \varepsilon$ for even larger n. So by Lemma (2.3), (g_n) has a subsequence that spans a complemented l_1 in L_F^*.

We now show that (h_n) has equi-absolutely continuous norms. Note that if $m \le n$ then $[\,|h_m| \ge a_n\,] = \emptyset$ while if $m > n$ then

$$
\begin{aligned}
\int_{[\,|h_m|\ge a_n\,]} F(h_m) d\mu &= \int_{[\,|f_m|<a_m\,]\cap[\,|f_m|\ge a_n\,]} F(f_m) d\mu \\
&= \int_{[\,|f_m|\ge a_n\,]} F(f_m) d\mu - \int_{[\,|f_m|\ge a_m\,]} F(f_m) d\mu \\
&\le \sup\{\int_{[\,|f|\ge a_n\,]} F(f) d\mu \;:\; f \in B_X\} - \delta + \frac{1}{m} \\
&\le \delta + \frac{1}{n} - \delta + \frac{1}{n} \\
&= \frac{2}{n}\,.
\end{aligned}
$$

So for each positive integer n we have

$$
\sup_m \int_{[\,|h_m|\ge a_n\,]} F(h_m) d\mu = \sup_{m>n} \int_{[\,|h_m|\ge a_n\,]} F(h_m) d\mu \le \frac{2}{n}\,.
$$

It follows then that $\{F(h_m) \;:\; m \ge 1\}$ is uniformly integrable in L^1 and so by Theorem (1.8), (h_n) has equi-absolutely continuous norms as we claimed. Hence by Theorem (1.9), (h_n) is relatively weakly compact in L_F^*. So by passing to appropriate subsequences, we can assume that (g_n) spans a complemented l_1 in L_F^* and (h_n) is weakly convergent in L_F^*. Thus $(h_{2n} - h_{2n+1})$ is weakly null. So by Mazur's theorem, there is an increasing sequence (n_k) of positive integers and a sequence (a_k) of non-negative reals so that

- $\sum_{j=n_k+1}^{n_{k+1}} a_j = 1$.
- The sequence (w_k) defined by $w_k = \sum_{j=n_k+1}^{n_{k+1}} a_j (h_{2j} - h_{2j+1})$ is norm-null in L_F^*.

Let

$$
u_k = \sum_{j=n_k+1}^{n_{k+1}} a_j (f_{2j} - f_{2j+1})
$$

and

$$v_k = \sum_{j=n_k+1}^{n_{k+1}} a_j(g_{2j} - g_{2j+1}) \ .$$

Then $u_k = v_k + w_k$ and $\|u_k - v_k\|_F = \|w_k\|_F \to 0$ as $k \to \infty$. By selection, $(\frac{g_n}{\|g_n\|_F})$ was equivalent to l_1's unit vector basis with complemented span in L_F^*. As $\|g_n\|_F \geq \int F(g_n)d\mu \geq \delta - \frac{1}{n}$, (g_n) itself is equivalent to l_1's unit vector basis. A little thought convinces us that this is also the case with (v_k), with the closed linear span of (v_k) still complemented in L_F^* of course. By passing to a subsequence to ensure that $\|u_k - v_k\|_F$ converges to zero fast enough to apply Lemma (2.2), the result is finished. ∎

2.2 Subspaces of L_F^* that have type

The work of Kadec and Pelczýnski in [4], finds its natural continuation in the work of Rosenthal. In [12], Rosenthal shows that a subspace of L^1 is reflexive if and only if it has non-trivial type. In this section, we follow his lead, to show that the same fact holds true for the special class of Orlicz spaces, we have been considering. The following result, mentioned in the form of a lemma, is due to Dor and Kauffman (see [3]).

Lemma 2.5 *Suppose* $f_1, \ldots, f_n \ \in B_{L^1(\mu)}$ *satisfy*

$$\|\sum_{i=1}^n a_i f_i\|_1 \geq \theta \sum_{i=1}^n |a_i| \ .$$

for any $a_1, \ldots, a_n$*, where* $0 < \theta < 1$.

Then there exist pairwise disjoint measurable sets $A_1, \ldots, A_n$ *such that*

$$\int_{A_i} |f_i| d\mu \geq \theta^2 \ .$$

We now adapt that lemma to our purposes.

Lemma 2.6 *Suppose* $f_1, \ldots, f_n \ \in B_{L_F^*(\mu)}$ *satisfy*

$$\|\sum_{i=1}^n a_i f_i\|_F \geq \theta \sum_{i=1}^n |a_i| \ ,$$

for any $a_1, \ldots, a_n$*, where* $0 < \theta < 1$. *Then there exist pairwise disjoint measurable sets* $A_1, \ldots, A_n$ *such that*

$$\|\chi_{A_i} f_i\|_F \geq \theta^2 \ .$$

Proof : There is no loss in assuming that $\|\sum_{i=1}^n a_i f_i\|_F > \theta \sum_{i=1}^n |a_i|$, provided that not all of $a_1, \ldots, a_n$ are zero. Choose now $g \in B_{L_G^*}$, where G is the complement of F, so that

$$|\int g(\sum_{i=1}^n a_i f_i) d\mu| > \theta \sum_{i=1}^n |a_i| \ .$$

Then

$$\int |\sum_{i=1}^n a_i (g f_i)| d\mu > \theta \sum_{i=1}^n |a_i|$$

and so by Lemma (2.5) there is a collection of measurable and pairwise disjoint sets $A_1, \ldots, A_n$ so that

$$\int_{A_i} |g f_i| d\mu \geq \theta^2 \quad \forall i = 1, \ldots, n \ .$$

By Hölder's inequality we then have that for each $i = 1, \ldots, n$

$$\begin{aligned} \|\chi_{A_i} f_i\|_F &\geq \|g\|_G \cdot \|\chi_{A_i} f_i\|_F \\ &\geq \int_{A_i} |g f_i| d\mu \\ &\geq \theta^2 \ , \end{aligned}$$

which is what we wanted. ∎

The following theorem, characterizes reflexive subspaces of L_F^*, for $F \in \Delta_2$, with complement G satisfying $\lim_{t\to\infty} \frac{G(mt)}{G(t)} = \infty$

Theorem 2.7 *Let $F \in \Delta_2$, with its complement G satisfying*

$$\lim_{t\to\infty} \frac{G(mt)}{G(t)} = \infty$$

for some $m > 0$. Let X be a subspace of L_F^*. Then the following are equivalent :

1. *The space X is not reflexive.*

2. *The space X contains a copy of l_1 complemented in L_F^*.*

3. *The space X contains l_1^n's uniformly.*

4. *The space X fails to have non-trivial type.*

Proof : The implication "1 ⇒ 2" is just theorem (2.4). As for "2 ⇒ 3" it follows directly from the definitions. The double implication "3 ⇔ 4" is Pisier's theorem (Theorem 1.10). So we will only show "3 ⇒ 1".

Suppose that X contains l_1^n's uniformly. Then there is a $0 < \theta < 1$ so that for each positive integer n, there are functions $f_1, \ldots, f_n \in B_X$ satisfying

$$\| \sum_{i=1}^{n} a_i f_i \|_F \geq \theta \sum_{i=1}^{n} |a_i| \ ,$$

for any choice of scalars $a_1, \ldots, a_n$. So by Lemma (2.6), we have that for each positive integer n, there are functions $f_1, \ldots, f_n \in B_X$ and measurable, pairwise disjoint sets $A_1, \ldots, A_n$ so that

$$\| \chi_{A_i} f_i \|_F \geq \theta^2 \quad i = 1, \ldots, n \ .$$

Since $A_1, \ldots, A_n$ are pairwise disjoint, at least one of them must have μ-measure less than $\frac{1}{n}$. Thus B_X cannot have equi-absolutely continuous norms. Hence by Theorem (1.9), B_X is not weakly compact in L_F^* and so X is not reflexive. ∎

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